UTRGV Open Mathematics Competition Fall 2024

Answer format: unless otherwise stated, all answers must be simplified fully. For trigonometric functions, exact values should be given for arguments that are integer multiples of $\pi/12$; inverse trigonometric functions whose values are integer multiples of $\pi/12$ should be evaluated exactly; logarithms and exponentials should be evaluated exactly when possible; fractions should be in lowest terms; radical expressions should be in simplest form, e.g., $3\sqrt{6}$ is okay but $\sqrt{54}$ isn't; and answers should include proper units where applicable. Multiples/powers of transcendental numbers can left as is, e.g. -5π and e^2 are okay, and ratios of radicals need not be rationalized, e.g., $\frac{8}{\sqrt{23}}$ is acceptable.

(1) Place the fraction $\frac{28}{64}$ in lowest terms.

(2) A semi-circle has a perimeter of 8 cm. Calculate its area.

(3) Find the integer n that minimizes $4n^2 - 106n + 12$.

(4) It costs *at least* \$1.50 for three apples and three oranges. It costs *at most* \$0.80 for an apple and 4 oranges. Assuming the items are priced per item and type only (no bulk discounts, etc.) and that all prices are nonnegative, determine...

- the smallest possible and largest possible prices for a single apple
- the smallest possible and largest possible prices for a single orange

(5) Let ABC be a triangle. Let D be a point on the side BC such that $\overline{AC} = \overline{CD}$ and $\overline{AD} = \overline{DB}$. If the angle $\widehat{B}$ is 11 times $\widehat{C}$, find $\widehat{A}$ in degrees.

(6) Find the smallest positive integer $n \neq 1$, which satisfies the following congruences simultaneously: $n \equiv 1 \pmod 4, n \equiv 1 \pmod 8$ and $n \equiv 1 \pmod{12}$.

(7) Five friends go to the theater to watch a movie. When they go to find their seats they find only six seats are left. How many different ways can the friends sit in the available seats?

(8) Compute the following integral

$$\int_0^{2\pi} \cos(x)\cos(2x)\cos(3x)\mathrm{d}x.$$

(9) Find the value of

$$\sum_{n=1}^{\infty} \frac{3^{-n}}{n}.$$

(10) Compute the value of the following continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}.$$

(11) Find a 2×2 matrix A with real entries such that

$$A^{101} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(12) Consider the ODE

$$x^2 y'''(x) + 7xy''(x) + 7y'(x) - \frac{7}{x} y(x) = 0.$$

Find the smallest $p \in \mathbb{R}$ so that $\overline{\lim}_{x \to 0^+} (y(x)x^p) < \infty$ for all solutions y. Here, $\overline{\lim}$ denotes the limit superior — it is also denoted lim sup.

(13) Suppose that r_1, r_2 are real numbers such that $r_1 r_2 = 2$. If r_1 and r_2 are roots of

$$x^4 - x^3 + ax^2 - 8x - 8 = 0,$$

find a.

(14) Evaluate the complex contour integral $\oint_C \frac{\sin z}{(z - \pi/4)^4} \mathrm{d}z$ where C is the closed curve $|z - \pi/4| = \frac{1}{2024}$ oriented *clockwise*.

(15) Provide an accurate approximation to the roots of

$$\log x - x + 1.02 = 0.$$

The roots you find should be accurate to within an error tolerance on the order of ~ 0.01. Here log is the natural logarithm.