UTRGV Open Mathematics Competition 2024

(1) Calculate the median of the following list of numbers: 12, 5, 13, 14, 0, -2, 0, 8.

(2) Find the greatest common divisor of 710 and 68.

(3) The Hailstone Sequence is defined recursively. Given a value a_n, the next term is computed via

$$a_{n+1} = \begin{cases} a_n/2, & a_n \text{ even} \\ 3a_n + 1, & a_n \text{ odd}. \end{cases}$$

Starting with $a_0 = 11$, find the first n so that $a_n = 1$.

(4) Suppose $f(x) = a + bx$ and that

$$f(f(f(0))) = 4$$
$$f(f(f(1))) = 68.$$

Evaluate $f(4/21)$.

(5) If $\sin x + \cos x = \frac{\sqrt{6}}{2}$, find the value of $\sin^4 x + \cos^4 x$.

(6) Let $\{x_n\}$ be a sequence of numbers that satisfy the following conditions:

$$x_1 + x_2 + ... + x_{n-1} + x_n = n^2 x_n$$

and $x_{99} = \frac{1}{9900}$. Find x_{100}.

(7) Find all real numbers m and n such that the graph of the function

$$f(x) = \sqrt{4x^2 + mx} - nx$$

has the horizontal asymptote $y = 1$ as $x \to \infty$.

(8) Find all values of p so that the following series converges

$$\sum_{n=2}^{\infty}\left(\frac{n^2}{\sqrt[3]{n^p-n}}+\left(\frac{1}{2}\right)^{n(20-p)}\right).$$

(9) Let V be the region in $\mathbb{R}^3$ bounded by the planes $x=0$, $y=0$, $z=0$, and $x+2y+3z=3$. Calculate

$$\int\int\int_V (3-x)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

(10) Find all solutions x_1, x_2, x_3, and y to the system of equations

$$\begin{cases} x_2-x_3=yx_1, \\ x_1-x_2=yx_2, \\ -x_1-x_3=yx_3, \end{cases}$$

where $x_3=1$.

(11) Find the remainder upon dividing 2^{350} by 34.

(12) Let $y(t)$ satisfy the differential equation

$$y''(t)=\frac{1}{(1+y)^2}$$

with $y(0)=0$, and $y'(0)=8$. Evaluate $\lim_{t\to\infty} y'(t)$.

(13) Find the polynomial $P(x)$ satisfying the functional equation

$$(x+1)P(x)=(x-4)P(x+1)$$

where $P(6)=1$.